Proof complexity of a CSP dichotomy proof

Azza Gaysin[1]

Passau university, Department of Mathematical Logic, Dr. Hans-Kapfinger Straße 30, 94032, Passau
azza.gaysin@uni-passau.de

Abstract. The constraint satisfaction problem (CSP) can be formulated as a homomorphism problem between relational structures: given structure $\mathcal{A}$, for any structure $\mathcal{X}$, whether there exists a homomorphism from $\mathcal{X}$ to $\mathcal{A}$. It was conjectured for years that all problems of this kind over finite domains are divided into polynomial time and NP-complete problems, and the conjecture was proved in 2017 separately by Zhuk [1] and Bulatov [2].
Zhuk's algorithm solves any tractable CSP($\mathcal{A}$) in polynomial time. For satisfiable instances, the algorithm produces a solution, i.e. a polynomial-size witness of an affirmative answer that one can independently check in polynomial time. That is not the case for unsatisfiable instances.
We use some proof complexity methods (formalization in theories of bounded arithmetic, propositional translations, etc.) to show that the algorithm may be appended to provide an independent proof of the correctness of the algorithm for negative answers too. We present the formalization of the algorithm in the theory of bounded arithmetic W_1^1 introduced in [3]. The formalization shows that W_1^1 proves the soundness of Zhuk's algorithm, where by soundness we mean that any rejection of the algorithm is correct. Together with the known relation of the theory to propositional calculus G, it follows that tautologies, expressing the non-existence of a solution for unsatisfiable instances, have short proofs in G.

Keywords: Constraint satisfaction problems · Bounded arithmetic · Proof complexity.

References

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